Control Lyapunov function method:

- Consider control affine system

$$\dot{x} = f(x) + g(x)u$$

- $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $g(x) \in \mathbb{R}^{n\times m}$

- $f(0) = 0$, $g(0) = 0$

- Objective: design feedback control law

  $u = k(x)$ to stabilize at $x = 0$ (AS or GAS)

Example:

① $\dot{x} = x^2 + xu$, $x, u \in \mathbb{R}$

Stabilizing feedback?

$u = k(x) = -(x+1) \implies \dot{x} = x^2 - x(x+1) = -x$

GAS

or if we want $K(0)=0$ (Control input is zero at eqlb.)

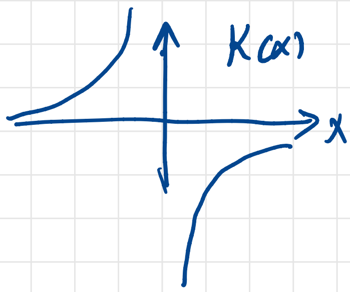
$$u = K(x) = -x - x^2$$

$$\Rightarrow \dot{x} = x^2 - x(x+x^2) = -x^3$$

GAS

② $\dot{x} = x + x^2 u$

$$u = K(x) = \begin{cases} -\frac{2}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \Rightarrow \dot{x} = -x \quad \text{GAS}$$



$\longrightarrow$ Not continuous at zero

Infact, it is impossible to stabilize with a continuous feedback law

① x^2 is small near $x=0$ so u should be large

② $u<0$ if $x>0$ and $u>0$ when $x<0$

③ $\dot{x} = x + x^2(x-1)u$

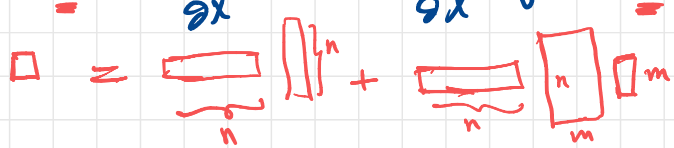
We can not have feedback that gives global stability because at $x=1$ we have no control.

- Design is simple in scalar examples we just look at the sign of RHS

- In general, design can be achived with Lyapunov functions.

$$(*) \qquad \dot{x} = f(x) + g(x)u$$

$$\dot{V}(x,u) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)\, u$$



- We want to choose $u = k(x)$ so that $\dot{V} < 0$

Def: a C^1 positive def. function $V(x)$ is called Control Lyapunov funct (CLF) for (*) if

$$\inf_u \dot{V}(x,u) < 0 \quad \forall x \neq 0$$

it means that

$$\forall x \neq 0, \ \exists u \text{ s.t. } \dot{V}(x,u) < 0$$

- Let's look at $\dot{V}(x,u)$ more closely

$$\dot{V}(x,u) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) u$$

$$= \frac{\partial V}{\partial x} f(x) + \sum_{i=1}^{m} \left(\frac{\partial V}{\partial x} g(x)\right)_i u_i$$

- For each x, as long as $\left(\frac{\partial V}{\partial x} g(x)\right)_i \neq 0$ for some i, we can choose u_i large enough to make $\dot{V} < 0$

- If $\frac{\partial V}{\partial x} g(x) = 0$ for some x, then we must have $\frac{\partial V}{\partial x} f(x) < 0$

Lemma: V is a CLF for $(*)$ iff

for all $\bar{x} \neq 0$ s.t. $\frac{\partial V}{\partial x} g(\bar{x}) = 0$, we have $\frac{\partial V}{\partial x} f(\bar{x}) < 0$

Def: V satisfies the small control property (SCP)

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\inf_{\|u\| \leq \varepsilon} \dot{V}(x,u) < 0 \quad \forall \|x\| \leq \delta$$

- It means that control is small when x is small.

- SCP captures additional requirement that $K(x)$ is continuous at $x = 0$

Thm: Suppose (#) has a CLF $V(x)$. Then, there exists a AS control law $k(x)$ which is C^1 away from $x=0$

- If V satisfies SCP, then $k(x)$ can be chosen to be continuous at $x=0$.

proof: (Sontag)

- gives explicit formula for control

- Let $L_f V = \underbrace{\frac{\partial V}{\partial x} f}_{\text{scalar } \mathbb{R}}$ and $L_g V = \underbrace{\frac{\partial V}{\partial x} g}_{\text{row vector } \mathbb{R}^m}$

$$k(x) = \begin{cases} -\dfrac{L_f V + \sqrt{(L_f V)^2 + \|L_g V\|^4}}{\|L_g V\|^2} (L_g V)^T & \text{if } L_g V \neq 0 \\ 0 & \text{if } L_g V = 0 \end{cases}$$

$$\dot{V} = L_f V + L_g V\, k(x)$$

$$= L_f V - \frac{1}{\cancel{\|L_g V\|^2}} \left(L_f V + \sqrt{(L_f V)^2 + \|L_g V\|^4} \right) \cancel{(L_g V)^T L_g V}$$

$$\Rightarrow \quad \dot{V} = -\sqrt{(L_f V)^2 + \|L_g V\|^4} < 0$$

because when $L_g V = 0$, we have $L_f V \neq 0$
(by lemma)

- The form of the law $k(x)$ is to ensure that it is C^1 away from $x = 0$
(smooth control law)

Example:

$$\dot{x} = x^2 + x u$$

- Take $V(x) = \frac{1}{2} x^2$

$$\Rightarrow \quad \dot{V}(x, u) = \underbrace{x^3}_{L_f V} + \underbrace{x^2}_{L_f g} u$$

- V is CLF because $L_g V \neq 0$ for all $x \neq 0$

$$k(x) = -\frac{x^3 + \sqrt{x^6 + x^8}}{x^4} x^2 = -x - |x| \sqrt{1 + x^2}$$

closed-loop: $\dot{x} = -x\,|x|\sqrt{1 + x^2}$

Optimal control interpretation of Sontag formula

- Let $a = L_f V$, $B = L_g V$ and $b = \|L_g V\|^2$
- We like to have $\dot{V} = a + Bu \leq 0$
- Consider the optimal control problem

$$\min \int_0^\infty (bz^2 + w^2)\,dt$$

$$\text{s.t.} \quad \dot{z} = az + Bw$$

- optimal control $w = -B^T p z$

where p solves Ricatti eq.

$$bp^2 - 2ap - b = 0 \implies p = \frac{a + \sqrt{a^2 + b^2}}{b}$$

- With optimal control law, $\dot{z} = az - BB^T p z$ is stable $\implies a - BB^T p < 0$

$\implies$ we can take $u = -B^T p$ to make

$\dot{V} = a + Bu < 0 \implies$ Sontag formula